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Impulsive effect on an elastic solid with generalized thermodiffusion

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Abstract The theory of generalized thermoelastic diffusion with one relaxation time is employed to study the distribution of temperature, displacement components, stresses, concentration and chemical potential in a semi-infinite medium having an impulsive mechanical load at the origin. Using the joint Laplace and Fourier transforms, the governing equations are transformed into a vector–matrix differential equation which is then solved by the eigenvalue approach. The solution of the problem in the physical domain is obtained numerically using a numerical method for the inversion of the Laplace and Fourier transforms. Results of this work are presented graphically and are compared with the results of generalized thermoelasticity and classical elasticity deduced as special cases.

Keywords Generalized thermoelasticity · Laplace and Fourier transforms · Mechanical load · Thermoelastic diffusion

1 Introduction

Duhamel [1] and Neumann [2] introduced the theory of uncoupled thermoelasticity. There are two shortcomings of this theory. Firstly, the fact that the mechanical state of the elastic body has no effect on the temperature is not in accordance with true physical experiments. Secondly, the heat equation being parabolic predicts an infinite speed of propagation for the temperature, which is physically inadmissible. Biot [3] developed the coupled theory of thermoelasticity which eliminates the first defect, but shares the second defect of uncoupled theory. In the classical theory of thermoelasticity, when an elastic solid is subjected to a thermal disturbance, the effect is felt at a location far from the load, instantaneously. This implies that the thermal wave propagates with infinite speed, a physically impossible result.

During the last four decades, wide-spread attention has been given to the thermoelasticity theories which admit a finite speed for the propagation of a thermal field. Lord and Shulman [4] reported a new theory based on the modified Fourier's Law of heat conduction with one relaxation time. This non-classical theory eliminates the paradox of

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infinite velocity of heat propagation and is termed as the generalized dynamical theory of thermoelasticity. Various problems characterizing this theory have been investigated and reveal some interesting phenomena. For this theory, Ignaczak [5] studied uniqueness of solution and Sherief [6] proved uniqueness and stability.

Modern approaches to the analytical treatment of nonclassical dynamical thermoelasticity have been described by Hetnarski and Ignaczak [7] by reviewing five different models of a thermoelastic solid in which disturbances are transmitted in a wave-like manner. The domain-of-influence theorem for a mixed displacement-temperature problem of L–S theory as proved by Ignaczak et al. [8] and Ignaczak [9] has been used by them to describe the phenomenon of propagation of a thermoelastic wave in the L–S model, showing that a thermoelastic disturbance propagates as a wave from the given domain with a finite speed due to the applied thermomechanical load. Similar results were also proved for four other models. Anwar and Sherief [10] and Sherief [11] developed the state-space approach to this theory. Anwar and Sherief [12] completed the integral-equation formulation. Sherief and Hamza [13] and Sherief and Hamza [14] solved some two-dimensional problems and studied wave propagation. Sherief and El-Maghraby [15] solved a problem for an internal penny-shaped crack. A detailed study of thermoelastic plane waves was made in [16–18]. A two-dimensional problem for a generalized thermoelastic half-space (L–S model) subjected to the effects of a thermal shock has been considered by Sherief and Helmy [19].

Lamb [20] was the first to investigate a disturbance generated in a semi-infinite elastic medium by an impulsive force applied along a line or at a point on the surface or inside the medium. Fung and Tong [21, Chap. 8] studied the problem of a line load suddenly applied on the surface of semi-infinite body of homogeneous isotropic linear elastic material. The disturbance due to mechanical point loads and thermal sources acting on the boundary of a homogeneous isotropic thermoelastic half-space has been investigated by Sharma and Chauhan [22] in the context of generalized theories of thermoelasticity. El-Maghraby [23] solved a two-dimensional problem in generalized thermoelasticity with heat sources.

Diffusion can be defined as the spontaneous migration of substances from regions of high concentration to regions with low concentration. There is now a great deal of interest in the study of this phenomenon, due to its many applications in geophysics and industrial applications. The phenomenon of diffusion is used to improve the conditions of oil extractions (seeking ways of more efficiently recovering oil from oil deposits). These days, oil companies are interested in the process of thermoelastic diffusion for more efficient extraction of oil from oil deposits.

Nowacki [24–27] developed the theory of thermoelastic diffusion. In this theory, a coupled thermoelastic model is used. This implies infinite speeds of propagation of thermoelastic waves. Recently, Sherief et al. [28] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. The present study is motivated by the importance of thermoelastic diffusion processes in the field of oil extraction.

A list of symbols used in this paper is given in the Appendix.

2 Basic equations and problem formulation

Following Sherief et al. [28], the governing equations for an isotropic, homogeneous elastic solid with generalized thermodiffusion at uniform temperature T_0 in the undisturbed state, in the absence of body forces and heat loads are:

(i) the equation of motion

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \beta_1 \Theta,_i - \beta_2 c,_i, \tag{1}$$

(ii) the generalized energy equation

$$K\Theta,_{ii} = \rho C_E(\dot{\Theta} + \tau_0 \ddot{\Theta}) + \beta_1 T_0(\dot{e_{kk}} + \tau_0 \ddot{e_{kk}}) + a T_0(\dot{c} + \tau_0 \ddot{c}), \tag{2}$$

(iii) the generalized diffusion equation

$$D\beta_2 e_{kk},_{ii} + Da\Theta,_{ii} + \dot{c} + \tau \ddot{c} - Dbc,_{ii} = 0, \tag{3}$$

(iv) the constitutive equations

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}(\lambda e_{kk} - \beta_1\Theta - \beta_2 c), \tag{4}$$

$$\zeta = -\beta_2 e_{kk} + bc - a\Theta, \tag{5}$$

where τ_0, the thermal relaxation time, ensures that the heat-conduction equation satisfied by temperature Θ predicts a finite speed of heat propagation and τ, the diffusion relaxation time, ensures that the equation satisfied by the concentration c also predicts a finite speed of propagation of matter from one medium to the other. The superposed dot denotes the derivative with respect to time.

We use a fixed Cartesian coordinate system (x, y, z) with origin on the surface $z = 0$, which is stress-free and with the z-axis directed vertically into the medium. The region $z > 0$ is occupied by an elastic solid with generalized thermodiffusion. A mechanical (normal or tangential) load of magnitude F_0 is assumed to be acting at a point on the surface $z = 0$ of the medium.

We restrict our analysis parallel to the xz-plane. The boundary of the medium is assumed to be thermally insulated. The chemical potential is also assumed to be a known function of time. We shall use the following non-dimensional variables

$$\begin{aligned}
&x^* = \frac{\omega}{c_1}x, \quad z^* = \frac{\omega}{c_1}z, \quad t^* = \omega t, \quad \tau^* = \omega\tau, \quad \tau_0^* = \omega\tau_0,\\
&u_x^* = \frac{\rho\omega c_1}{\beta_1 T_0}u_x, \quad u_z^* = \frac{\rho\omega c_1}{\beta_1 T_0}u_z, \quad \sigma_{ij}^* = \frac{\sigma_{ij}}{\beta_1 T_0},\\
&c^* = \frac{c}{C_0}, \quad \zeta^* = \frac{\zeta}{P_0}, \quad \Theta^* = \frac{\Theta}{T_0},
\end{aligned} \tag{6}$$

where

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad \omega = \frac{\rho C_E c_1^2}{K}. \tag{7}$$

Using the quantities given by (6) in (1)–(3), we obtain the equations in dimensionless form (dropping the asterisks for convenience) as

$$\frac{\partial^2 u_x}{\partial x^2} + a_1\frac{\partial^2 u_z}{\partial x\partial z} + a_2\frac{\partial^2 u_x}{\partial z^2} - \frac{\partial\Theta}{\partial x} - a_3\frac{\partial c}{\partial x} - \frac{\partial^2 u_x}{\partial t^2} = 0, \tag{8}$$

$$a_2\frac{\partial^2 u_z}{\partial x^2} + a_1\frac{\partial^2 u_x}{\partial x\partial z} + \frac{\partial^2 u_z}{\partial z^2} - \frac{\partial\Theta}{\partial z} - a_3\frac{\partial c}{\partial z} - \frac{\partial^2 u_z}{\partial t^2} = 0, \tag{9}$$

$$\tau_m\frac{\partial\Theta}{\partial t} + b_1\tau_m\frac{\partial}{\partial t}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + b_2\tau_m\frac{\partial c}{\partial t} - b_3\nabla^2\Theta = 0, \tag{10}$$

$$\frac{\partial}{\partial x}\left(\nabla^2 u_x\right) + \frac{\partial}{\partial z}\left(\nabla^2 u_z\right) + b_4\nabla^2\Theta + b_5\tau_n\frac{\partial c}{\partial t} - b_6\nabla^2 c = 0, \tag{11}$$

where

$$\begin{aligned}
&a_1 = \frac{\lambda + \mu}{\lambda + 2\mu}, \quad a_2 = \frac{\mu}{\lambda + 2\mu}, \quad a_3 = \frac{\beta_2 C_0}{\beta_1 T_0}, \quad b_1 = \frac{\beta_1^2 T_0}{\rho^2 c_1^2 C_E},\\
&b_2 = \frac{aC_0}{C_E\rho}, \quad b_3 = \frac{K\omega}{\rho c_1^2 C_E}, \quad b_4 = \frac{a\rho c_1^2}{\beta_1\beta_2}, \quad b_5 = \frac{\rho C_0 c_1^4}{D\beta_1\beta_2 T_0\omega},\\
&b_6 = \frac{b\rho C_0 c_1^2}{\beta_1\beta_2 T_0}, \quad \tau_m = \left(1 + \tau_0\frac{\partial}{\partial t}\right), \quad \tau_n = \left(1 + \tau\frac{\partial}{\partial t}\right),\\
&\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.
\end{aligned} \tag{12}$$

With the aid of the expressions relating displacement components u_x, u_z to the scalar potential ϕ and vector potential ψ in dimensionless form given by

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \quad u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{13}$$

in (8)–(11), we obtain

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2}\right]\phi - \Theta - a_3 c = 0, \tag{14}$$

$$\nabla^2 \psi - \frac{1}{a_2}\frac{\partial^2 \psi}{\partial t^2} = 0, \tag{15}$$

$$\left[\nabla^2 - \frac{1}{b_3}\tau_m \frac{\partial}{\partial t}\right]\Theta - \frac{b_1}{b_3}\tau_m \frac{\partial}{\partial t}\nabla^2 \phi - \frac{b_2}{b_3}\tau_m \frac{\partial c}{\partial t} = 0, \tag{16}$$

$$\nabla^2 \phi + b_4 \nabla^2 \Theta + \left[b_5 \tau_n \frac{\partial}{\partial t} - b_6 \nabla^2\right] c = 0. \tag{17}$$

3 Solution of the problem

3.1 Formulation of a vector–matrix differential equation in the transformed domain

We now apply the Laplace and Fourier transforms defined by

$$\hat{f}(x, z, p) = \int_0^\infty f(x, z, t)\mathrm{e}^{-pt}\mathrm{d}t, \tag{18}$$

$$\tilde{f}(\xi, z, p) = \int_{-\infty}^{\infty} \hat{f}(x, z, p)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}x, \tag{19}$$

where p and ξ are the Laplace- and Fourier-transform variables respectively, so that (14)–(17) reduce to the form

$$\frac{\mathrm{d}^2\tilde{\phi}}{\mathrm{d}z^2} = R_{11}\tilde{\phi} + R_{12}\tilde{\Theta} + R_{13}\tilde{c}, \tag{20}$$

$$\frac{\mathrm{d}^2\tilde{\Theta}}{\mathrm{d}z^2} = R_{21}\tilde{\phi} + R_{22}\tilde{\Theta} + R_{23}\tilde{c}, \tag{21}$$

$$\frac{\mathrm{d}^2\tilde{C}}{\mathrm{d}z^2} = R_{31}\tilde{\phi} + R_{32}\tilde{\Theta} + R_{33}\tilde{c}, \tag{22}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \left(\xi^2 + \frac{p^2}{a_2}\right)\right]\tilde{\psi} = 0, \tag{23}$$

where

$$\begin{aligned}
&R_{11} = (p^2 + \xi^2), \quad R_{12} = 1, \quad R_{13} = a_3,\\
&R_{21} = f_1, \quad R_{22} = f_2, \quad R_{23} = f_3,\\
&R_{31} = \frac{g_1}{b_6 - a_3}, \quad R_{32} = \frac{g_2}{b_6 - a_3}, \quad R_{33} = \frac{g_3}{b_6 - a_3},\\
&g_1 = p^4 + f_1(1 + b_4),\\
&g_2 = p^2 + (f_2 - \xi^2)(1 + b_4),\\
&g_3 = a_3(p^2 - \xi^2) + f_3(1 + b_4) + b_5\tau_n^* p + b_6\xi^2,\\
&f_1 = \frac{b_1}{b_3}\tau_m^* p^3, \quad f_2 = \frac{(1 + b_1)}{b_3}\tau_m^* p + \xi^2, \quad f_3 = \frac{(b_1 a_3 + b_2)}{b_3}\tau_m^* p.\\
&\tau_m^* = 1 + \tau_0 p, \quad \tau_n^* = 1 + \tau p.
\end{aligned} \tag{24}$$

The system of equations (20)–(22) can be written in the form of a vector–matrix differential equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}z}V(\xi, z, p) = A(\xi, p)V(\xi, z, p), \tag{25}$$

where

$$V = \begin{bmatrix} U \\ D^*U \end{bmatrix}, \quad A = \begin{bmatrix} O & I \\ A_1 & O \end{bmatrix}, \quad U = \begin{bmatrix} \tilde{\phi} \\ \tilde{\Theta} \\ \tilde{c} \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}, \tag{26}$$

where D^* denotes the differentiation with respect to z, i.e., $\mathrm{d}/\mathrm{d}z$.

3.2 Solution of the vector–matrix differential equation

We now proceed to solve Eq. 25 by an eigenvalue approach. To solve (25), we take

$$V(\xi, z, p) = X(\xi, p)\mathrm{e}^{qz}, \tag{27}$$

so that

$$A(\xi, p)V(\xi, z, p) = qV(\xi, z, p), \tag{28}$$

which leads to an eigenvalue problem. The characteristic equation corresponding to the matrix A is given by

$$\det[A - qI] = 0, \tag{29}$$

which on expansion gives

$$q^6 - \lambda_1 q^4 + \lambda_2 q^2 - \lambda_3 = 0, \tag{30}$$

where

$$\begin{aligned} \lambda_1 &= R_{11} + R_{22} + R_{33}, \\ \lambda_2 &= R_{11}R_{22} + R_{22}R_{33} + R_{33}R_{11} - R_{12}R_{21} - R_{23}R_{32} - R_{31}R_{13}, \\ \lambda_3 &= R_{11}(R_{22}R_{33} - R_{23}R_{32}) + R_{12}(R_{23}R_{31} - R_{21}R_{33}) + R_{13}(R_{21}R_{32} - R_{22}R_{31}). \end{aligned} \tag{31}$$

The roots of Eq. 30, which are the eigenvalues of the matrix A, are $\pm q_i, \; i = 1, 2, 3$. We assume that the real parts of q_i are positive. The eigenvector $X(\xi, p)$ corresponding to the eigenvalues q_i can be determined by solving the homogeneous equation

$$[A - qI]X(\xi, p) = 0. \tag{32}$$

The set of eigenvectors $X_i(\xi, p)$, $(i = 1, 2, 3, 5, 6, 7)$ may be obtained as

$$X_i(\xi, p) = \begin{bmatrix} X_{i1}(\xi, p) \\ X_{i2}(\xi, p) \end{bmatrix}, \tag{33}$$

where

$$X_{i1}(\xi, p) = \begin{bmatrix} s_i \\ r_i \\ 1 \end{bmatrix}, \quad X_{i2}(\xi, p) = \begin{bmatrix} s_i q_i \\ r_i q_i \\ q_i \end{bmatrix},$$

$$q = q_i; \; i = 1, 2, 3$$

$$X_{j1}(\xi, p) = \begin{bmatrix} s_i \\ r_i \\ 1 \end{bmatrix}, \quad X_{j2}(\xi, p) = \begin{bmatrix} -s_i q_i \\ -r_i q_i \\ -q_i \end{bmatrix},$$

$$j = i + 4, \quad q = -q_i; \; i = 1, 2, 3 \tag{34}$$

$$s_i = \frac{s_{i1} - s_{i2} - R_{23}s_{i3}}{R_{21}s_{i3}},$$

$$r_i = \frac{R_{31}R_{13} - (R_{33} - q_i^2)(R_{11} - q_i^2)}{R_{32}(R_{11} - q_i^2) - R_{12}R_{31}},$$

$$s_{i1} = (R_{11} - q_i^2)(R_{22} - q_i^2)(R_{33} - q_i^2),$$

$$s_{i2} = R_{31}R_{13}(R_{22} - q_i^2),$$

$$s_{i3} = (R_{32}(R_{11} - q_i^2) - R_{12}R_{31}); \quad i = 1, 2, 3.$$

The solution of (25) is given by

$$V(\xi, z, p) = \sum_{i=1}^{3} [B_i X_i(\xi, p) e^{q_i z} + B_{i+4} X_{i+4}(\xi, p) e^{-q_i z}] \tag{35}$$

and the solution of (23) is

$$\tilde{\psi} = B_4 e^{q_4 z} + B_8 e^{-q_4 z}, \tag{36}$$

where $B_i (i = 1, 2, 3, 4, 5, 6, 7, 8)$ are arbitrary constants and

$$q_4 = \sqrt{\xi^2 + \frac{p^2}{a_2}}. \tag{37}$$

Equations (35) and (36) represent the solution of the general problem in the case of generalized thermodiffusion elasticity by employing the eigenvalue approach and therefore can be applied to a broad class of problems in the domain of Laplace and Fourier transforms.

4 Application: interactions due to a mechanical load

In this section, the general solutions for displacement, stresses, temperature field, deviation in concentration and chemical potential given by (35) and (36) will be used to yield the response of a half-space subjected to an impulsive mechanical load. The constants B_i will be determined by imposing the proper boundary conditions. These constants, when substituted in Eqs. 35 and 36, enable us to obtain the required physical quantities in the Fourier- and Laplace-transformed (ξ, z, p) domain. The final solution in the original domain (x, z, t) is obtained by a numerical inversion of both transforms.

4.1 Case 1: Load in the normal direction

In the half-space, the load $F(x)$ is applied in the normal direction at the origin of the co-ordinate system. The boundary $z = 0$ is assumed to be thermally insulated so that there is no variation of temperature and concentration on it. Therefore, for this loading case, the boundary conditions are

$$\sigma_{zz} = -F(x)\delta(t), \quad \sigma_{zx} = 0, \quad \frac{\partial \Theta}{\partial z} = 0, \quad \frac{\partial c}{\partial z} = 0, \quad at\ z = 0, \tag{38}$$

where $F(x) = F_0 \delta(x)$.

4.2 Case 2: Load in the tangential direction

In the half-space, the load $F(x)$ is applied in the tangential direction at the origin of the co-ordinate system. The boundary conditions in this case are

$$\sigma_{zz}=0,\quad \sigma_{zx}=-F(x)\delta(t),\quad \frac{\partial\Theta}{\partial z}=0,\quad \frac{\partial c}{\partial z}=0,\quad at\ z=0. \tag{39}$$

It can be seen that eight unknowns are to be determined in (35) and (36) and only four boundary conditions appear in each case. For the half-space the radiation conditions imply outgoing waves with decreasing amplitudes in the positive z-direction. Therefore, the radiation conditions require that $B_1=B_2=B_3=B_4=0$.

We obtain the expressions for the displacement components, stresses, temperature field, concentration and potential as

$$\tilde{\sigma_{zz}}=e_1B_5\mathrm{e}^{-q_1z}+e_2B_6\mathrm{e}^{-q_2z}+e_3B_7\mathrm{e}^{-q_3z}+e_4B_8\mathrm{e}^{-q_4z}, \tag{40}$$
$$\tilde{\sigma_{zx}}=k_1B_5\mathrm{e}^{-q_1z}+k_2B_6\mathrm{e}^{-q_2z}+k_3B_7\mathrm{e}^{-q_3z}-k_4B_8\mathrm{e}^{-q_4z}, \tag{41}$$
$$\tilde{u_x}=-\mathrm{i}\xi[s_1B_5\mathrm{e}^{-q_1z}+s_2B_6\mathrm{e}^{-q_2z}+s_3B_7\mathrm{e}^{-q_3z}]+q_4B_8\mathrm{e}^{-q_4z}, \tag{42}$$
$$\tilde{u_z}=-[s_1q_1B_5\mathrm{e}^{-q_1z}+s_2q_2B_6\mathrm{e}^{-q_2z}+s_3q_3B_7\mathrm{e}^{-q_3z}]-\mathrm{i}\xi B_8\mathrm{e}^{-q_4z}, \tag{43}$$
$$\tilde{\Theta}=r_1B_5\mathrm{e}^{-q_1z}+r_2B_6\mathrm{e}^{-q_2z}+r_3B_7\mathrm{e}^{-q_3z}, \tag{44}$$
$$\tilde{c}=B_5\mathrm{e}^{-q_1z}+B_6\mathrm{e}^{-q_2z}+B_7\mathrm{e}^{-q_3z}, \tag{45}$$
$$\tilde{\xi}=M_1B_5\mathrm{e}^{-q_1z}+M_2B_6\mathrm{e}^{-q_2z}+M_3B_7\mathrm{e}^{-q_3z}, \tag{46}$$

where

$$B_{i+4}=\frac{\Delta_i}{\Delta},\quad i=1,2,3,4,$$
$$M_i=-e^*(q_i^2-\xi^2)s_i+\frac{bC_0}{P_0}-\frac{aT_0r_i}{P_0},\quad i=1,2,3,$$

$$\Delta=(r_3-r_2)q_2q_3(e_1k_4+e_4k_1)+(r_1-r_3)q_1q_3(e_2k_4+e_4k_2)+(r_2-r_1)q_1q_2(e_3k_4+e_4k_3), \tag{47}$$
$$e_i=q_i^2s_i-a^*s_i-r_i-b^*;\quad i=1,2,3,\quad e_4=\mathrm{i}\xi q_4\left(1-\frac{\lambda}{\rho c_1^2}\right),$$
$$k_i=\frac{2\mathrm{i}\xi s_iq_i\mu}{\rho c_1^2};\quad i=1,2,3,\quad k_4=\frac{\mu}{\rho c_1^2}(\xi^2+q_4^2),$$
$$e^*=\frac{\beta_1\beta_2T_0}{\rho c_1^2P_0},\quad a^*=\frac{\lambda\xi^2}{\rho c_1^2},\quad b^*=\frac{\beta_2C_0}{\beta_1T_0}.$$

4.3 Case 1: In the normal direction

The values of $\Delta_i;\ i=1,2,3,4,$ when the load is acting in the normal direction, are

$$\Delta_1=F_0k_4(r_2-r_3)q_2q_3,\quad \Delta_2=F_0k_4(r_3-r_1)q_3q_1,\quad \Delta_3=F_0k_4(r_1-r_2)q_1q_2,$$
$$\Delta_4=F_0\left[k_1(r_2-r_3)q_2q_3+k_2(r_3-r_1)q_1q_3+k_3(r_1-r_2)q_1q_2\right]. \tag{48}$$

4.4 Case 2: In the tangential direction

The solution for this case is as in (40)–(46), only with the replacement of $\Delta_i;\ i=1,2,3,4$ as

$$\begin{aligned}&\Delta_1=F_0e_4(r_2-r_3)q_2q_3,\quad \Delta_2=F_0e_4(r_3-r_1)q_3q_1,\quad \Delta_3=F_0e_4(r_1-r_2)q_1q_2,\\&\Delta_4=-F_0[e_1(r_2-r_3)q_2q_3+e_2(r_3-r_1)q_1q_3+e_3(r_1-r_2)q_1q_2].\end{aligned} \tag{49}$$

4.5 Particular case I

By taking $c = D = a = b = \beta_2 = 0$, we obtain the expressions for the displacement components, stresses and temperature field in the generalized thermoelastic medium as:

$$\tilde{\sigma_{zz}} = e_1^* B_4^* \mathrm{e}^{-q_1^* z} + e_2^* B_5^* \mathrm{e}^{-q_2^* z} + e_3^* B_6^* \mathrm{e}^{-q_3^* z}, \tag{50}$$

$$\tilde{\sigma_{zx}} = k_1^* B_4^* \mathrm{e}^{-q_1^* z} + k_2^* B_5^* \mathrm{e}^{-q_2^* z} - k_3^* B_6^* \mathrm{e}^{-q_3^* z}, \tag{51}$$

$$\tilde{u_x} = -\mathrm{i}\xi [s_1^* B_4^* \mathrm{e}^{-q_1^* z} + s_2^* B_5^* \mathrm{e}^{-q_2^* z}] + q_3^* B_6^* \mathrm{e}^{-q_3^* z}, \tag{52}$$

$$\tilde{u_z} = -[s_1^* q_1^* B_4^* \mathrm{e}^{-q_1^* z} + s_2^* q_2^* B_5^* \mathrm{e}^{-q_2^* z}] - \mathrm{i}\xi B_6^* \mathrm{e}^{-q_3^* z}, \tag{53}$$

$$\tilde{\Theta} = B_4^* \mathrm{e}^{-q_1^* z} + B_5^* \mathrm{e}^{-q_2^* z}, \tag{54}$$

where

$$q_i^{*2} = \frac{\lambda_1^* + (-1)^{i+1}\sqrt{\lambda_1^{*2} - 4\lambda_2^*}}{2}; \quad i = 1, 2, \tag{55}$$

are the roots of the equation

$$q^4 - \lambda_1^* q^2 + \lambda_2^* = 0, \tag{56}$$

where

$$\begin{aligned}
&\lambda_1^* = R_{11} + R_{22}, \quad \lambda_2^* = R_{11}R_{22} - R_{21}R_{12}, \\
&q_3^* = q_4^2, \quad B_{i+3}^* = \Delta_i^*/\Delta^*; \ i = 1, 2, 3\ , \\
&\Delta^* = q_1^*(e_2^* k_3^* + e_3^* k_2^*) - q_2^*(e_3^* k_1^* + e_1^* k_3^*), \\
&e_i^* = q_i^{*2} s_i^* - a^* s_i^* - 1; \ i = 1, 2, \quad e_3^* = \mathrm{i}\xi q_3^* \left(1 - \tfrac{\lambda}{\rho c_1^2}\right), \\
&k_i^* = \tfrac{2\mu}{\rho c_1^2}\left(\mathrm{i}\xi q_i^* s_i^*\right); \ i = 1, 2, \quad k_3^* = \tfrac{\mu}{\rho c_1^2}\left(q_3^{*2} + \xi^2\right), \\
&s_i^* = -\tfrac{R_{22} - q_i^{*2}}{R_{21}}; \ i = 1, 2.
\end{aligned} \tag{57}$$

4.5.1 Case 1: In the normal direction

The values of $\Delta_i^*;\ i = 1, 2, 3$, when the load is in the normal direction, are

$$\Delta_1^* = F_0 q_2^* k_3^*, \quad \Delta_2^* = -F_0 q_1^* k_3^*, \quad \Delta_3^* = F_0\left[q_2^* k_1^* - q_1^* k_2^*\right]. \tag{58}$$

4.5.2 Case 2: In the tangential direction

The solutions for this case are as in (50)–(54) only with the replacement of $\Delta_i^*; i = 1, 2, 3$ as

$$\Delta_1^* = F_0 q_2^* e_3^*, \quad \Delta_2^* = -F_0 q_1^* e_3^*, \quad \Delta_3^* = F_0\left[q_1^* e_2^* - q_2^* e_1^*,\right]. \tag{59}$$

4.6 Particular case II

If we neglect the thermodiffusion effect from the medium considered, the corresponding expressions for the displacement components and stresses are given by:

$$\tilde{\sigma_{zz}} = e_1' B_3' \mathrm{e}^{-q_1' z} + e_2' B_4' \mathrm{e}^{-q_2' z}, \tag{60}$$

$$\tilde{\sigma_{zx}} = k_1' B_3' \mathrm{e}^{-q_1' z} - k_2' B_4' \mathrm{e}^{-q_2' z}, \tag{61}$$

$$\tilde{u_x} = -(\mathrm{i}\xi) B_3' \mathrm{e}^{-q_1' z} + q_2' B_4' \mathrm{e}^{-q_2' z}, \tag{62}$$

$$\tilde{u_z} = -q_1' B_3' \mathrm{e}^{-q_1' z} + \mathrm{i}\xi B_4' \mathrm{e}^{-q_2' z}, \tag{63}$$

where

$$q_1' = \sqrt{p^2+\xi^2}, \quad q_2' = \sqrt{\frac{p^2}{a_2}+\xi^2},$$
$$B_{i+2}' = \Delta_i'/\Delta'; \; i=1,2, \quad \Delta' = -(e_1'k_2' + e_2'k_1'),$$
$$e_1' = q_1'^2 - a^*, \quad e_2' = \mathrm{i}\xi q_2'\left(1-\frac{\lambda}{\rho c_1^2}\right),$$
$$k_1' = \frac{2\mu}{\rho c_1^2}(\mathrm{i}\xi q_1'), \quad k_2' = \frac{\mu}{\rho c_1^2}(\xi^2+q_2'^2). \tag{64}$$

4.6.1 Case 1: In the normal direction

The values of $\Delta_i'; \; i=1,2$, when the load is in the normal direction are

$$\Delta_1' = F_0k_2', \quad \Delta_2' = F_0k_1'. \tag{65}$$

4.6.2 Case 2: In the tangential direction

The solution for this case are as in (60)–(63), only with the replacement of $\Delta_i'; \; i=1,2$ as

$$\Delta_1' = F_0e_2', \quad \Delta_2' = -F_0e_1'. \tag{66}$$

5 Inversion of transforms

The transformed displacements, stresses, temperature field, concentration and chemical potential (40)–(46), (50)–(54) and (60)–(63) are functions of z, the parameters of Laplace and Fourier transforms p and ξ, respectively, and hence are of the form $\tilde{f}(\xi, z, p)$. To get the function $f(x, z, t)$ in the physical domain, first we invert the Fourier transform using

$$\hat{f}(x,z,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}\xi x}\tilde{f}(\xi,z,p)\,\mathrm{d}\xi = \frac{1}{\pi}\int_0^{\infty}\{\cos(\xi x)\tilde{f}_e - \mathrm{i}\sin(\xi x)\tilde{f}_o\}\,\mathrm{d}\xi, \tag{67}$$

where $\tilde{f}_e$ and $\tilde{f}_o$ denote the even and odd parts of the function $\tilde{f}(\xi, z, p)$, respectively. Thus, expression (67) gives us the Laplace transform $\hat{f}(x, z, p)$ of function $f(x, z, t)$. Following Honig and Hirdes [29], we can convert the Laplace transform function $\hat{f}(x, z, p)$ to $f(x, z, t)$.

The last step is to evaluate the integral in Eq. 67. The method for evaluating this integral is given in [30, Chap. 4]; it involves the use of Romberg's integration with adaptive step size. This also uses the results from a successive refinement of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

6 Numerical results and discussion

Copper was chosen for the purpose of numerical evaluations. The material constants of the problem are given by Thomas [31] in SI units as follows:

$$T_0 = 293\,\mathrm{K}, \quad \rho = 8954\,\mathrm{kg/m^3}, \quad \tau_0 = 0.02\,\mathrm{s}, \quad \tau = 0.2\mathrm{s},$$
$$C_E = 383.1\,\mathrm{J/(kg\ K)}, \quad \alpha_t = 1.78 \times 10^{-5}\mathrm{K}^{-1}, \quad K = 386\,\mathrm{W/(m\ K)},$$
$$\lambda = 7.76 \times 10^{10}\,\mathrm{kg/(m\ s^2)}, \quad \mu = 3.86 \times 10^{10}\,\mathrm{kg/(m\ s^2)},$$
$$\alpha_c = 1.98 \times 10^{-4}\,\mathrm{m^3/kg}, \quad D = 0.85 \times 10^{-8}\,\mathrm{kg\ s/m^3},$$
$$a = 1.2 \times 10^4\,\mathrm{m^2/(s^2\ K)}, \quad b = 0.9 \times 10^6\,\mathrm{m^5/(kg\ s^2)}.$$

Comparisons are given for the dimensionless normal displacement $u_z(= u_z/F_0)$, normal stress $\sigma_{zz}(= \sigma_{zz}/F_0)$, temperature $\Theta(= \Theta/F_0)$, concentration $c(= c/F_0)$ and chemical potential $\zeta(= \zeta/F_0)$ for three different cases: a solid with thermoelastic diffusion (THED), a thermoelastic solid (THE) and an elastic solid subjected to normal and tangential impulsive loads have been studied and shown in Figs. 1–10 . The computations are carried out for two values of the non-dimensional time, namely for $t = 0.05$ and $t = 0.10$ at $z = 1.0$; the initial concentration is $C_0 = 1$ and the initial potential $P_0 = 1$, in the range $0 \leq x \leq 20$.

6.1 Case I: normal load applied

The comparisons of the dimensionless normal displacement u_z, normal stress σ_{zz}, temperature Θ, deviation in mass concentration c and deviation in chemical potential ζ for the three different cases are studied in Figs. 1–5.

Figure 1 represents the variation of the normal displacement u_z with distance x. The values of the displacement u_z have been magnified by multiplying with 10 for the THE theory and by 100 for the elastic theory for both times to depict the comparsion simultaneously in same figure. The values of the displacement u_z for the THED and THE theories increase with distance, whereas for the case of the elastic theory, the response of the displacement with respect to distance x is reverse. Very near to the point of application of the source there is a great difference in magnitude of the normal displacement for all three considered media. The effect of diffusion can be observed clearly from this figure by comparing the curves for the THED and THE theories.

Figure 2 shows the variations of the normal stress σ_{zz} with x resulting from a normal load. The values of the normal stress for the THED theory have been multiplied by 10 to enable comparison in the same ranges. For both times, the values of σ_{zz} for the THED theory are greater than the corresponding values for thermoelastic and elastic solids in the initial range. The reverse variations of the normal stress for the THED theory to that for the THE and elastic theories are observed owing to the diffusion factor. The temperature variation with distance due to a normal load is observed in Fig. 3. The values of the temperature Θ for the THED theory have been magnified by multiplying with 10^2. This large difference in numerical values of the temperature Θ between the two theories is

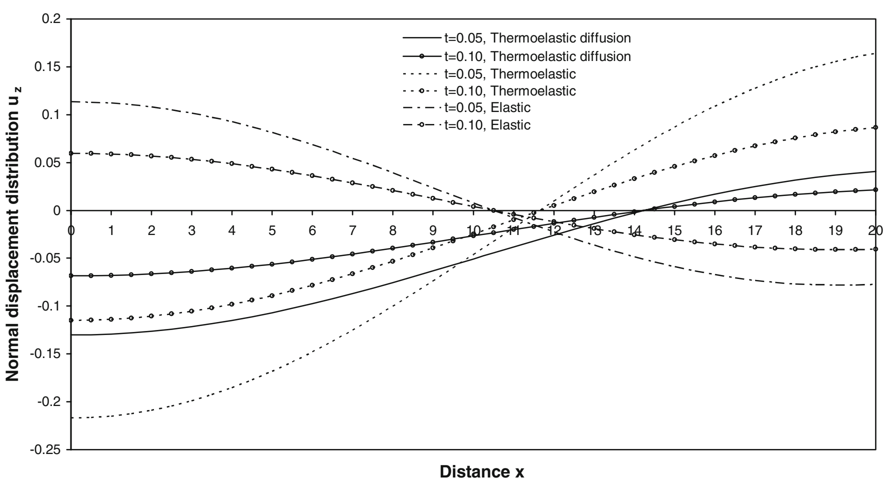

Fig. 1 Distribution of normal displacement u_z (due to normal load) versus distance

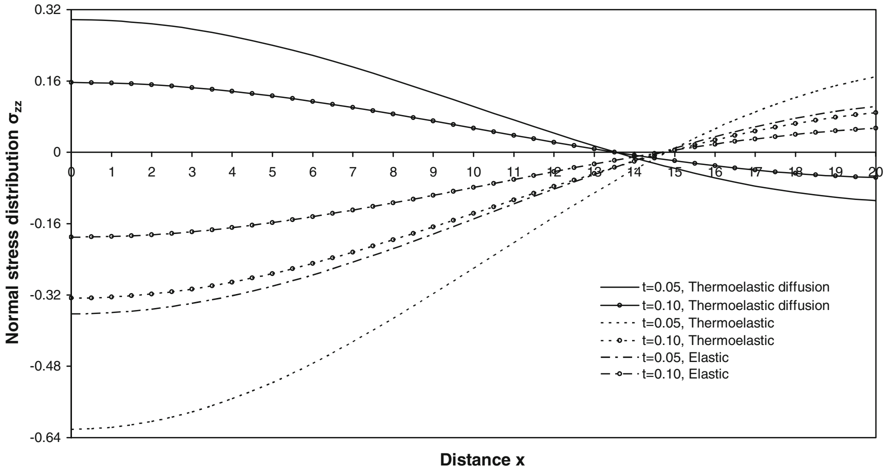

Fig. 2 Distribution of normal stress σ_{zz} (due to normal load) versus distance

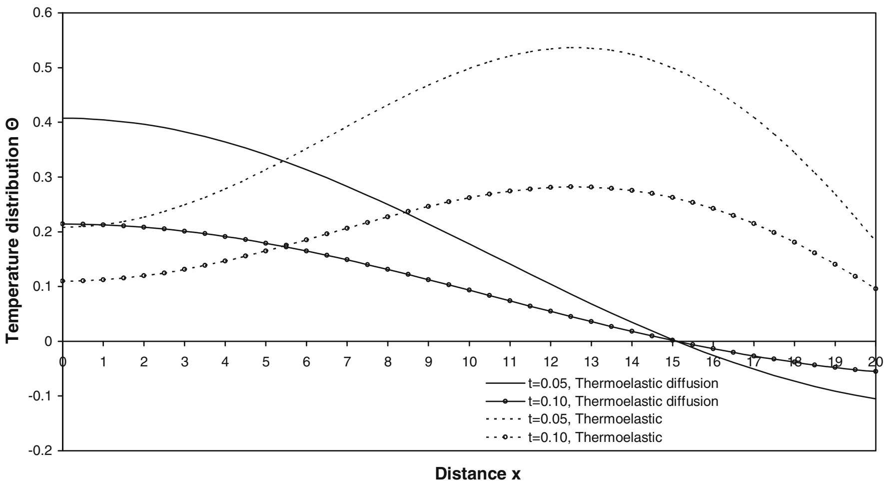

Fig. 3 Distribution of temperature Θ (due to normal load) versus distance

due to the presence of a concentration field in the THED theory. The magnitude of the temperature obtained from the THED theory seems to vanish earlier than that for the THE theory, where diffusion effects are absent.

The variation of the mean concentration about the initial concentration is represented by Fig. 4 for the THED theory. The difference in the values of c at a particular point for two different times can easily be observed from the graphs. It is also clearly depicted in the figure that the values of the mean concentration c are maximum at the origin for both times and, after a small oscillatory behaviour, seem to be vanishing far from origin. Figure 5 presents the distribution of the change in the chemical potential about the initial potential with distance x at both times. The values of the chemical potential for a particular range show sufficient difference for the two times.

6.2 Case II: tangential load applied

The comparisons of the dimensionless normal displacement u_z, normal stress σ_{zz}, temperature Θ, concentration deviation c and change in chemical-potential ζ for the three different cases are studied in Figs. 6–10.

Figure 6 shows the variations of the normal displacement u_z with x due to the application of a tangential load. The values of the displacement u_z for THE and elastic theories have been multiplied by 10^2 and 10^4, respectively.

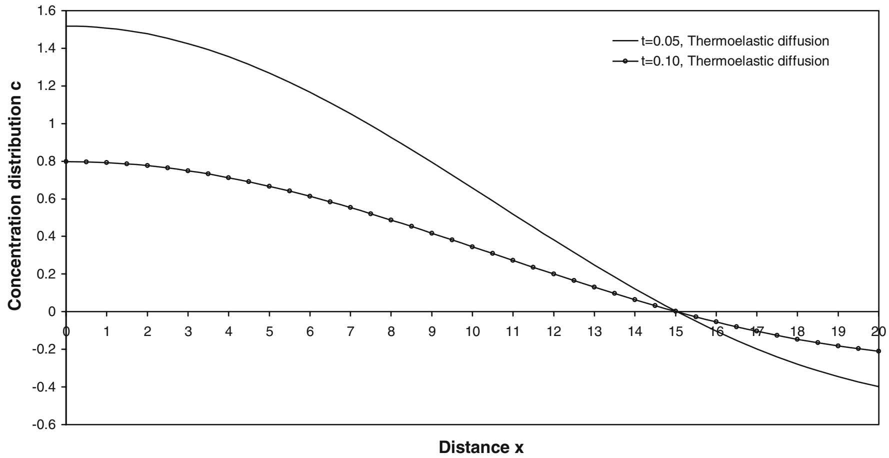

Fig. 4 Distribution of concentration c (due to normal load) versus distance

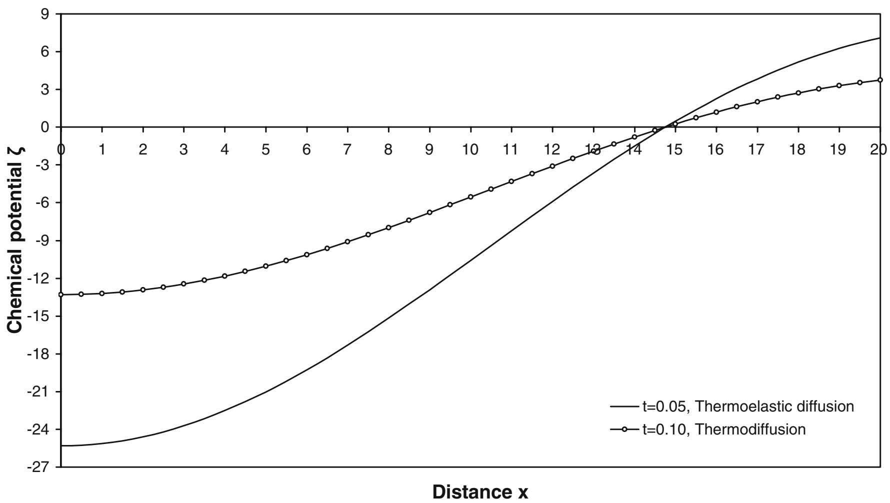

Fig. 5 Distribution of chemical potential ζ (due to normal load) versus

The trend of change in the displacement for the THED and elastic theories is opposite in nature to that for the THE theory for both times, whereas the behaviour of the variation with time are the same for all the three theories. Due to the influence of the chemical-mass term, the values of the normal displacement u_z are large in the THED medium in comparision with THE and elastic media. Figure 7 depicts the variations of the normal stress σ_{zz} with distance x due to the application of a tangential load. The values of the normal stress for the THE theory have been magnified by multiplying with 10 for both times and for the elastic theory the values are multiplied by 10^2 for both times. The behaviour of the variations of σ_{zz} for the THED theory is similar to that due to a normal load in Fig. 2. As the value of x increases, the values of the normal stress approach zero for all cases .

Figure 8 presents the variations of the temperature with distance x. The behaviour of the variations of the temperature for the THED theory is opposite in nature to that for the THE theory owing to the presence of the chemical-mass term. The deviation of the concentration from the mean value for the THED theory has been depicted in Fig. 9. The values of the concentration for a particular range show sufficient difference for the two times. Figure 10 shows the deviation of the chemical potential from the initial chemical potential with distance x

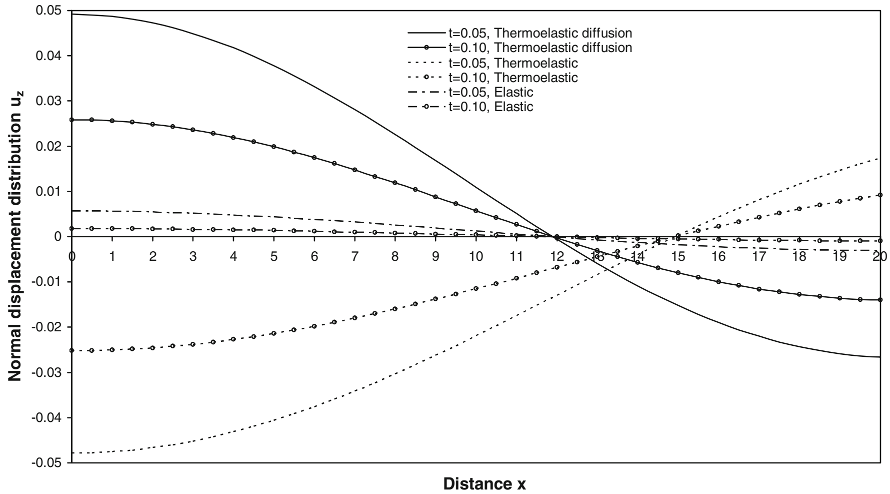

Fig. 6 Distribution of normal displacement u_z (due to tangential load) versus distance

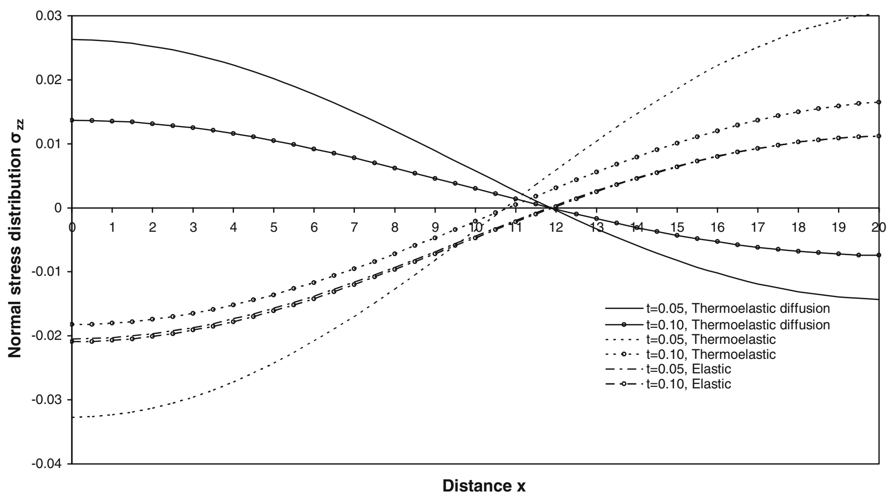

Fig. 7 Distribution of normal stress σ_{zz} (due to tangential load) versus distance

for the THED theory for both times. The values of ζ are maximum at the origin for both times and are decreasing smoothly thereafter in the further range.

7 Conclusion

The work presented in this article provides a mathematical model to obtain the two-dimensional solutions of temperature, displacements, stresses, mass concentration and chemical potential due to an impulsive mechanical load in a semi-infinite medium in the context of generalized thermodiffusion elasticity. An eigenvalue approach is used, which has the advantage of finding the solution of equations in the coupled form directly in matrix notation. It is evident from the figures that the effect due to a mechanical load on a generalized thermoelastic medium with diffusion depends upon the distance x.

Variations in the various physical quantites are observed to be quite significant at small times and near the vicinity of the loads/sources and remain close to zero afterwards. This is so because the free surface is subjected to an instantaneous source. This establishes the transient behaviour of the waves. Thus, the effect of the loading

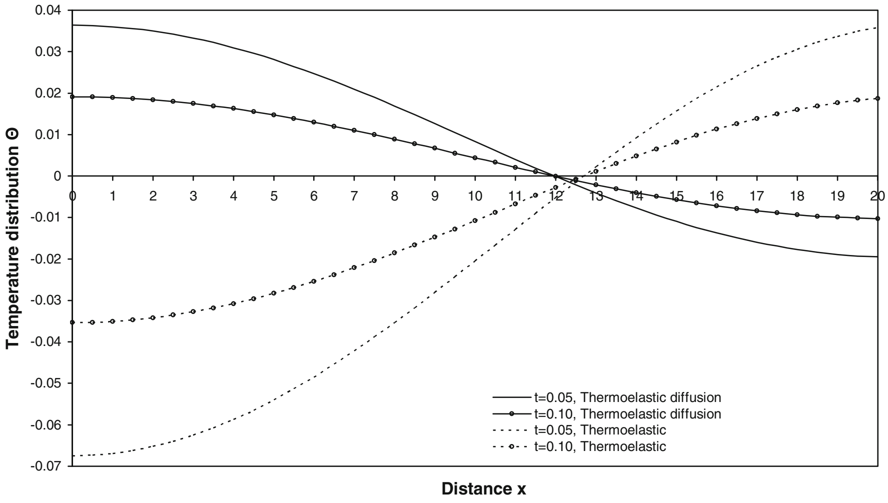

Fig. 8 Distribution of temperature Θ (due to tangential load) versus distance

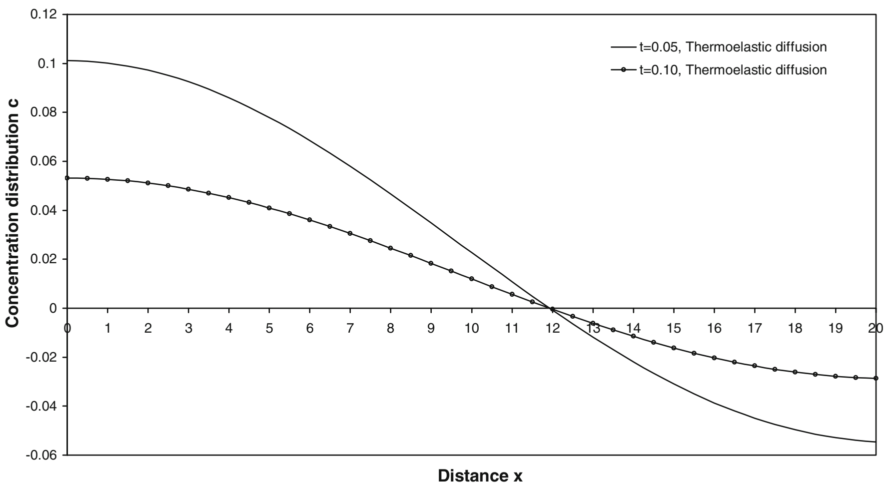

Fig. 9 Distribution of concentration c (due to tangential load) versus distance

does not reach infinity instantaneously but remains in a bounded domain that expands with the passage of time, as demanded by the L–S theory of thermoelasticity. Significant effects of the chemical-mass term, diffusion and thermal parameters on the normal displacement, stress and temperature are observed for two values of the time as depicted in the figures.

The analysis of the normal displacement, normal-stress component, temperature, mean concentration and chemical-potential deviation generated in a body due to an impulsive mechanical load (normal and tangiential) is an interesting mechanical problem with applications in determining the stability of a medium. Using these results, it is possible to investigate the disturbance caused by more general sources for practical applications. The present theoretical results may provide interesting information for experimental scientists/researchers/seismologists working on this subject. The introduction of diffusion parameters to the generalized thermoelastic medium provides a more realistic model for these studies. The methodology used in the present article is applicable to a wide range of problems in thermodynamics.

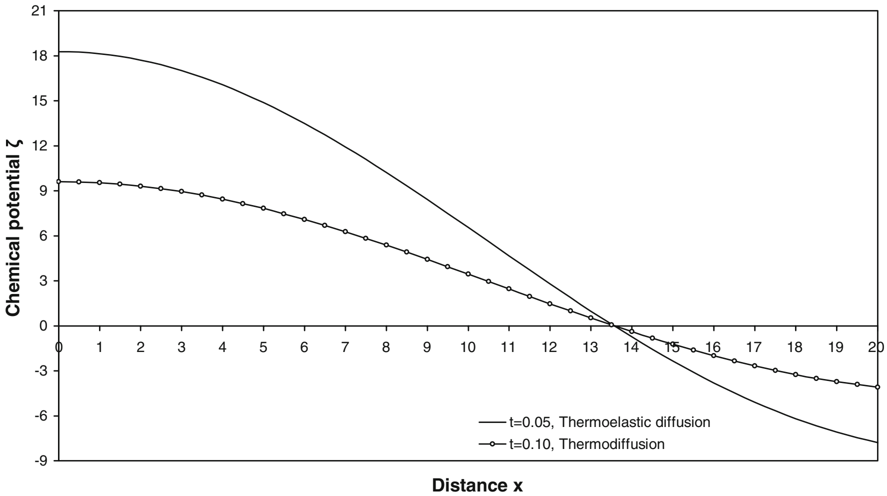

Fig. 10 Distribution of chemical potential ζ (due to tangential load) versus distance

Appendix. Nomenclature

λ, μ	Lame's constants
σ_{ij}	components of stress tensor
u_i	components of displacement vector
C_E	specific heat at constant strain
T_0	reference temperature chosen so that $\frac{\lvert T-T_0\rvert}{T_0} \ll 1$
K	thermal conductivity
P	chemical potential per unit mass at non equilibrium conditions
P_0	chemical potential per unit mass of natural state
C_0	mass concentration at natural state
c	$C - C_0$
τ_0	thermal-relaxation time
a	measure of thermodiffusion effect
β_1	$(3\lambda + 2\mu)\alpha_t$,
α_t	coefficient of linear thermal expansion
F_0	intensity of the applied mechanical load
ϕ	scalar potential
$\delta(.)$	Dirac delta function.
ρ	density of the medium
e_{ij}	components of strain tensor
t	time
T	absolute temperature
Θ	$T - T_0$
e_{kk}	dilatation
δ_{ij}	Kronecker delta
ζ	$P - P_0$
C	non-equilibrium concentration
D	thermodiffusion constant
τ	diffusion-relaxation time

b measure of diffusive effects
β_2 $(3\lambda + 2\mu)\alpha_c$
α_c coefficient of linear diffusion expansion
$\mathbf{u}$ displacement vector
ψ vector potential

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